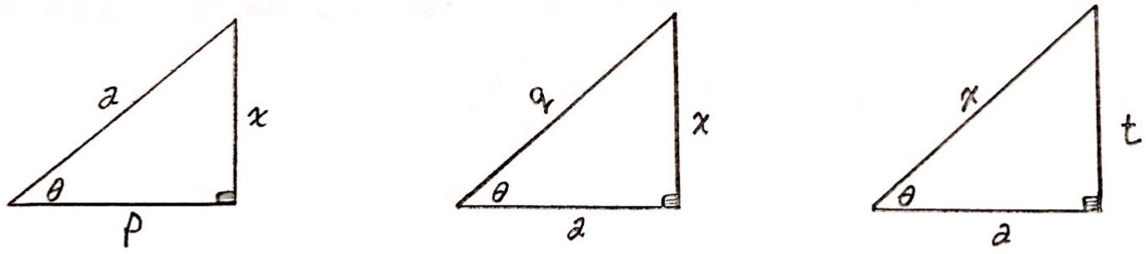
Some trigonometrical and hyperbolic substitutions:

⇒ We can evaluate integrals involving the radicals $\sqrt{x^2-a^2}$, $\sqrt{a^2-x^2}$ and $\sqrt{a^2+x^2}$ by using the following substitutions:

for integral involving	let	Then
1) $p=\sqrt{a^2-x^2}$	$x=a\sin\theta$	$p=a\cos\theta$
2) $q=\sqrt{a^2+x^2}$	$x=a\tan\theta$	$q=a\sec\theta$
3) $t=\sqrt{x^2-a^2}$	$x=a\sec\theta$	$t=a\tan\theta$



(1) (2) (3)

(الدالة المتكاملة)

⇒ This method aims to eleminate the radicals (الجذور) in the integrand hence The integrand takes Fundamental formula.

Note

* $\cos^2\theta+\sin^2\theta=1$.
* $1+\tan^2\theta=\sec^2\theta$.
* $\sec^2\theta-1=\tan^2\theta$.

Evaluate:

(1) $I=\int\frac{dx}{\sqrt{a^2-x^2}}$

Ans: let $x=a\sin\theta$, Then $dx=a\cos\theta\,d\theta$, Then

$$I=\int\frac{a\cos\theta}{(a^2-a^2\sin^2\theta)^{0.5}}d\theta=\int\frac{a\cos\theta}{a\cos\theta}d\theta=\int d\theta=\theta+C$$

$$=\boxed{\sin^{-1}\left(\frac{x}{a}\right)+C}$$

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(2) $J = \int \frac{dx}{x^2 \cdot \sqrt{a^2 - x^2}}$

Ans: Put $x = a \sin\theta \Rightarrow \sqrt{a^2 - x^2} = a\sqrt{1 - \sin^2\theta} = a \cos\theta$,

$dx = a \cos\theta \, d\theta$, Then

$$J = \int \frac{\cancel{a \cos\theta}}{a^2 \sin^2\theta \cdot \cancel{a \cos\theta}} d\theta = \frac{1}{a^2} \int \csc^2\theta \, d\theta = -\frac{1}{a^2} \cot\theta + C$$

$$= \boxed{\frac{-1}{a^2} \cot\left[\sin^{-1}\frac{x}{a}\right] + C} \quad ※$$

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(3) $K = \int \sqrt{a^2 - x^2} \, dx$

Ans: Put $x = a \sin\theta \Rightarrow \sqrt{a^2 - x^2} = a \cos\theta$, $dx = a \cos\theta \, d\theta$, Then

$$K = \int a \cos\theta \cdot a \cos\theta \, d\theta = a^2 \int \cos^2\theta \, d\theta = \frac{a^2}{2} \int [\cos 2\theta + 1] \, d\theta$$

$$= \frac{a^2}{2} \left[\frac{1}{2} \sin 2\theta + \theta \right] + C$$

$$= \frac{a^2}{4} \sin 2\theta + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + C \quad , \quad \sin 2\theta = 2 \sin\theta \cos\theta$$

$$= 2 \cdot \frac{x}{a} \cdot \frac{1}{a} \cdot \sqrt{a^2 - x^2}$$

$$\therefore \boxed{K = \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2} \sqrt{a^2 - x^2} + C}$$

(4) $L = \int \frac{dx}{a^2 - x^2}$

Ans: Put $x = a \sin\theta \Rightarrow a^2 - x^2 = a^2 \cos^2\theta$, $dx = a \cos\theta \, d\theta$

$$\therefore L = \int \frac{\cancel{a \cos\theta}}{a^2 \cos^{\cancel{2}}\theta} d\theta = \frac{1}{a} \int \sec\theta \, d\theta * \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta} \text{, Then}$$

$$= \frac{1}{a} \int \frac{\sec^2\theta + \sec\theta \tan\theta}{\tan\theta + \sec\theta} d\theta$$

Put $u = \tan\theta + \sec\theta \Rightarrow du = (\sec^2\theta + \sec\theta \tan\theta) \, d\theta$

$$\therefore L = \frac{1}{a} \int \frac{du}{u} = \frac{1}{a} \ln|u| = \frac{1}{a} \ln|\sec\theta + \tan\theta| + C$$

انظر الخلف

$$= \boxed{\frac{1}{a} \ln\left| \sec\left[\sin^{-1}\left(\frac{x}{a}\right)\right] + \tan\left[\sin^{-1}\left(\frac{x}{a}\right)\right] \right| + C} \quad ※$$

(5) $I = \int \frac{dx}{\sqrt{a^2+x^2}}$

Ans: Put $x = a\tan\theta \Rightarrow dx = a\sec^2\theta\, d\theta$,

$a^2+x^2 = a^2 + a^2\tan^2\theta = a^2(1+\tan^2\theta) = a^2\sec^2\theta$, Then

$\sqrt{a^2+x^2} = a\sec\theta$, Then

$I = \int \frac{a\sec^2\theta}{a\sec\theta}\, d\theta = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C \rightarrow (1)$

and we have $\tan\theta = \frac{x}{a}$ and $\sec\theta = \frac{\sqrt{a^2+x^2}}{a}$ sub. in (1), Then

$I = \ln|x+\sqrt{a^2+x^2}| - \ln|a| + C = \boxed{\sinh^{-1}\frac{x}{a} + C}$

خدنا الحد $\ln|a|$ ثابت
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(6) $J = \int \frac{dx}{x^2 \cdot \sqrt{a^2+x^2}}$

Ans: Put $x = a\tan\theta \Rightarrow dx = a\sec^2\theta\, d\theta$, similar to Pervious Prob.

$a^2+x^2 = a^2\sec^2\theta \Rightarrow \sqrt{a^2+x^2} = a\sec\theta$, Then

$J = \int \frac{\cancel{a}\sec^2\theta\, d\theta}{a^2 \cdot \tan^2\theta \cdot \cancel{a\sec\theta}} = \frac{1}{a^2}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta = \frac{1}{a^2}\int \frac{\cos^{\cancel{2}}\theta}{\cancel{\cos\theta} \cdot \sin^2\theta}\, d\theta$

$= \frac{1}{a^2}\int \cos\theta \cdot \sin^{-2}\theta\, d\theta \Rightarrow$ Put $u = \sin\theta \Rightarrow du = \cos\theta\, d\theta$, Then

$J = \frac{1}{a^2}\int u^{-2}\, du = \frac{1}{a^2} \cdot -u^{-1} + C = \frac{-1}{a^2} \cdot \frac{1}{\sin\theta} + C$

$= \frac{-1}{a^2}\csc\theta + C \rightarrow (1)$, and $\rightarrow = \boxed{\frac{-1}{a}\csc\left[\tan^{-1}\frac{x}{a}\right] + C}$

We have $\tan\theta = \frac{x}{a} \Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{x}{a} \Rightarrow \frac{\cos\theta}{\sin\theta} = \frac{a}{x}$,

$\therefore \cos\theta \cdot \csc\theta = \frac{a}{x} \rightarrow (2)$ and we have

$\sec\theta = \frac{\sqrt{a^2+x^2}}{a} \Rightarrow \cos\theta = \frac{a}{\sqrt{a^2+x^2}} \rightarrow$ sub in (2), Then

$\frac{\cancel{a}}{\sqrt{a^2+x^2}} \cdot \csc\theta = \frac{\cancel{a}}{x} \Rightarrow \csc\theta = \frac{\sqrt{a^2+x^2}}{x} \rightarrow$ sub. in (1), Then

$$\boxed{J = \frac{-\sqrt{a^2+x^2}}{a^2 \cdot x} + C}$$

(7) $K = \int \sqrt{a^2 + x^2}\, dx$

Ans: Put $x = a \tan\theta \Rightarrow dx = a \sec^2\theta\, d\theta$, also

$\sqrt{a^2 + x^2} = a \sec\theta$, Then

$K = \int a \sec\theta \cdot a \sec^2\theta\, d\theta = a^2 \int \sec^3\theta\, d\theta$ [will be solved using integration by part. (سوف يُشرح لاحقاً)]

$= \frac{1}{2} a^2 [\sec\theta \tan\theta + \ln|\sec\theta + \tan\theta|] + C \rightarrow ①$

and we have $\tan\theta = \frac{x}{a}$, $\sec\theta = \frac{\sqrt{a^2 + x^2}}{a}$ sub. in ①, we obtain

$K = \frac{1}{2} x \sqrt{a^2 + x^2} + \frac{1}{2} a^2 \ln|x + \sqrt{a^2 + x^2}| - \ln|a| + C$

$= \boxed{\frac{1}{2} x \sqrt{a^2 + x^2} + \frac{1}{2} a^2 \sinh^{-1} \frac{x}{a} + C'}$ #

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(8) $L = \int \frac{dx}{a^2 + x^2}$

Ans: Put $x = a \tan\theta \Rightarrow dx = a \sec^2\theta\, d\theta$, also

$a^2 + x^2 = a^2 \sec^2\theta$. Then

$L = \int \frac{\cancel{a \sec^2\theta}}{a^{\cancel{2}} \cancel{\sec^2\theta}} d\theta = \frac{1}{a} \int d\theta = \frac{1}{a}\theta + C = \boxed{\frac{1}{a} \tan^{-1} \frac{x}{a} + C}$

(9) $I = \int \frac{dx}{\sqrt{x^2 - a^2}}$

Ans: Put $x = a \sec\theta \Rightarrow dx = a \sec\theta \cdot \tan\theta \cdot d\theta$, also

$\sqrt{x^2 - a^2} = \sqrt{a^2 \sec^2\theta - a^2} = a\sqrt{\sec^2\theta - 1} = a\sqrt{\tan^2\theta} = a \tan\theta$, Then

$I = \int \frac{\cancel{a} \sec\theta\, \cancel{\tan\theta}}{\cancel{a \tan\theta}} d\theta = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$

and we have $\sec\theta = \frac{x}{a}$, $\tan\theta = \frac{\sqrt{x^2 - a^2}}{a}$, Then

$I = \ln|x + \sqrt{x^2 - a^2}| - \ln|a| + C$

$\therefore \boxed{I = \cosh^{-1} \frac{x}{a} + C'}$, where $C' = C - \ln|a|$.

(10) $J = \int \frac{dx}{x^2\sqrt{x^2-a^2}}$

Ans: Put $x = a\sec\theta \Rightarrow dx = a\sec\theta\tan\theta\, d\theta$, also

$\sqrt{x^2-a^2} = a\tan\theta$, Then

$J = \int \frac{\cancel{a\sec\theta\tan\theta}}{a^2\sec^{\cancel{2}}\theta\,\cancel{a\tan\theta}}\, d\theta = \frac{1}{a^2}\int \cos\theta\, d\theta = \frac{1}{a^2}\sin\theta + C, \rightarrow (1)$

we have $\frac{x}{a} = \sec\theta \Rightarrow \cos\theta = \frac{a}{x}$, also

$\frac{\sqrt{x^2-a^2}}{a} = \tan\theta = \frac{\sin\theta}{\cos\theta} \Rightarrow \sin\theta = \frac{a}{x}\cdot\frac{\sqrt{x^2-a^2}}{a} = \frac{\sqrt{x^2-a^2}}{x}$ sub. in (1)

$\therefore \boxed{J = \frac{\sqrt{x^2-a^2}}{a^2 x} + C}$

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(11) $K = \int \sqrt{x^2-a^2}\, dx$

Ans: Put $x = a\sec\theta \Rightarrow dx = a\sec\theta\tan\theta\, d\theta$, $\sqrt{x^2-a^2} = a\tan\theta$

$K = \int a\tan\theta \cdot a\sec\theta\tan\theta\, d\theta = \int a^2\tan^2\theta\sec\theta\, d\theta$

$= a^2\int \tan\theta\cdot\tan\theta\cdot\sec\theta\, d\theta$, $d(\sec\theta) = \sec\theta\cdot\tan\theta\cdot d\theta$, Then

$K = a^2\int \tan\theta\, d(\sec\theta) \overset{\text{how?}}{=} a^2\left[\sec\theta\cdot\tan\theta - \int \sec^3\theta\, d\theta\right]$

→ integration by Parts.
(أشرح لاحقا)

$= \frac{1}{2}a^2\left[\sec\theta\tan\theta - \ln|\sec\theta + \tan\theta|\right] + C$

$\frac{x}{a}$, $\frac{\sqrt{x^2-a^2}}{a}$, $\frac{x}{a}$, $\frac{\sqrt{x^2-a^2}}{a}$

$\boxed{K = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{1}{2}a^2\ln|x + \sqrt{x^2-a^2}| + C'}$

(12) $\int \frac{dx}{x^2-a^2} = L$

Ans: Put $x = a\sec\theta \Rightarrow dx = a\sec\theta\tan\theta\, d\theta$, $\sqrt{x^2-a^2} = a\tan\theta$. Then

$L = \int \frac{\cancel{a}\sec\theta\,\cancel{\tan\theta}}{a^{\cancel{2}}\tan^{\cancel{2}}\theta}\, d\theta = \frac{1}{a}\int \frac{\cancel{\cos\theta}}{\cancel{\cos\theta}\sin\theta}\, d\theta = \frac{1}{a}\int \csc\theta\, d\theta$

$= \frac{1}{a}\ln|\csc\theta - \cot\theta| + C$, $\cot\theta = \frac{1}{\tan\theta} = \frac{a}{\sqrt{x^2-a^2}}$,

$\cos\theta = \frac{1}{\sec\theta} = \frac{a}{x}$,

$= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{x^2-a^2}}{a} \Rightarrow \frac{1}{\sin\theta} = \csc\theta = \frac{x}{\sqrt{x^2-a^2}}$

$\cos\theta \to \frac{a}{x}$

$= \boxed{\frac{-1}{a}\coth^{-1}\frac{x}{a} + C}$

There are some of suggested substitution for certian types of integrals:

If you find in the integrand ⟶ Put

1. $\sqrt{a^2 - x^2} \longrightarrow x = a\sin\theta$
2. $\sqrt{a^2 + x^2} \longrightarrow x = a\tan\theta$
3. $\sqrt{x^2 - a^2} \longrightarrow x = a\sec\theta$

.. Expression containing fractional powers of x. ⟶ $x = z^r$, where r is least common multiple of the fractional indices (م.م.أ)

5. $x f(x^2) \longrightarrow x^2 = z.$

6. $f(x, \sqrt{ax+b}) \longrightarrow z = \sqrt{ax+b}.$
7. $f(x, \sqrt{(x-a)(x-b)}) \longrightarrow \sqrt{x-b} = z\sqrt{x-a}.$
8. $f(x, \sqrt{(b-x)(x-a)}) \longrightarrow \sqrt{b-x} = z\sqrt{x-a}.$
9. $f(x, \sqrt{x^2+bx+c}) \longrightarrow x + \sqrt{x^2+bx+c} = z.$
10. $(x^2+a^2)^n$, $n = +ve \longrightarrow x = a\sinh z.$
11. $(x^2+a^2)^n$, $n = -ve \longrightarrow x = a\tan\theta.$

متنوعة تعويضات

Miscellanous substitutions:

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1. $I = \int \frac{dx}{\sqrt{x} - \sqrt[3]{x}}$ ⟶ Common root, $u^6 = x$

Ans: Put $u = \sqrt[6]{x} \Rightarrow u = x^{\frac{1}{6}} \Rightarrow du = \frac{1}{6} x^{\frac{-5}{6}}\, dx$

$\boxed{\sqrt{x} = x^{\frac{1}{2}} = u^3}$, $\boxed{\sqrt[3]{x} = x^{\frac{1}{3}} = u^2}$, $du = \frac{1}{6} u^{-5}\, dx \Rightarrow \boxed{dx = 6u^5\, du}$

Then $I = \int \frac{6u^5}{u^3 - u^2}\, du = 6\int \frac{u^{5\,3}}{u^2(u-1)}\, du = 6\int \frac{u^3 - 1 + 1}{u-1}\, du$

$$= 6\left[\int \frac{\cancel{(u-1)}(u^2+u+1)}{\cancel{(u-1)}}\, du + \int \frac{du}{u-1}\right]$$

$$= 6\left[\int (u^2+u+1)\, du + \int \frac{du}{u-1}\right]$$

$$= 6\left[\frac{1}{3}u^3 + \frac{1}{2}u^2 + u + \ln|u-1|\right] + C$$

$$= \boxed{2\sqrt{x} + 3\sqrt[3]{x} + 6\sqrt[6]{x} + 6\ln|\sqrt[6]{x} - 1| + C}$$

2. $J = \int \frac{x^{3/4}}{\sqrt{x} - \sqrt[3]{x}} \, dx$

Ans:

$$J = \int \frac{\sqrt[4]{x^3}}{\sqrt{x} - \sqrt[3]{x}} \, dx \rightarrow ①$$

for such a Problem, you should find the Common root.

The Common root for $\sqrt{-}$, $\sqrt[3]{-}$, $\sqrt[4]{-}$ is $\sqrt[12]{-}$, Then

Put $u = \sqrt[12]{x} \Rightarrow x = u^{12} \Rightarrow \boxed{\sqrt{x} = u^6}$, $\boxed{\sqrt[3]{x} = u^4}$, $\boxed{\sqrt[4]{x^3} = u^9}$, also $\boxed{dx = 12\, u^{11} \cdot du}$, Then sub. in ①, so

$$J = \int \frac{u^9}{u^6 - u^4} \cdot 12\, u^{11} \, du = 12 \int \frac{u^{20}}{u^4 (u^2 - 1)} \, du = 12 \int \frac{u^{16}}{u^2 - 1} \, du$$

$$= 12 \int \frac{u^{16} - 1 + 1}{u^2 - 1} \, du = 12 \left[\int \frac{u^{16} - 1}{u^2 - 1} \, du + \int \frac{du}{u^2 - 1} \right]$$

using long dèvision ; $\frac{u^{16} - 1}{u^2 - 1} = u^8 + u^4 + u^2 + 1$

$\therefore J = \frac{12}{9} u^9 + \frac{12}{5} u^5 + \frac{12}{3} u^3 + 12\, u + 12 \coth^{-1}(u) + C$, or

$$\boxed{J = \frac{4}{3} x^{\frac{3}{4}} + \frac{12}{5} x^{\frac{5}{12}} + 4\, x^{\frac{1}{4}} + 12\, x^{\frac{1}{12}} + 12 \coth^{-1}\left(x^{1/12}\right) + C}$$

for integrals involving functions of the sine and cosine, use the substitution $\boxed{u = \tan \frac{x}{2}}$, also this substitution will work for rational integrands involving of the six trigonometric functions as any one of them can be written as rational Combinations of the sine and cosine.

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